

Bijection between Conjugacy Classes and Irreducible Representations of Finite Inverse Semigroups *

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Abstract

In this paper we show that the irreducible representations of a finite inverse semigroup S over an algebraically closed field F are in bijection with the conjugacy classes of S if the characteristic of F is zero or a prime number that does not divide the order of any maximal subgroup of S .

Keywords: Conjugacy, Irreducible representation, Finite inverse semigroup.

Mathematics Subject Classification 2010: 20M18, 20M32.

1 Introduction

A useful result in group representation theory is that there is a bijection between the conjugacy classes of a finite group G and the irreducible representations of G over an algebraically closed field F with characteristic not a factor of the order of $|G|$. Is this the same for the case of semigroup representation theory?

There are different conjugacy relations in semigroup theory. The following two are natural generalizations of the usual group conjugacy. Let S be a monoid with unit group G . Two elements $a, b \in S$ are G -conjugate, denoted by $a \sim_G b$, if there is $g \in G$ such that $b = gag^{-1}$. Let S be a semigroup. Then elements $a, b \in S$ are called *primarily S -conjugate*, denoted by $a \sim_p b$, if there are $x, y \in S$ for which $a = xy$ and $b = yx$. This relation is reflexive and symmetric, but not transitive. Its transitive closure is referred to as S -conjugacy and will be denoted by \sim . If S is a group, then \sim_G and \sim coincide with the usual group conjugacy. If S is a monoid and $a \sim_G b$ then $a \sim b$, in other words, \sim_G is finer than \sim .

Which conjugacy will lead to an affirmative answer to the question above? Not the \sim_G -conjugacy. Indeed, there are usually more \sim_G -conjugacy classes in a monoid than its irreducible representations over F . For example, the rook monoid of size 3 has 10 \sim_G -conjugacy classes but 7 irreducible representations (see [16, 33] for more details). The S -conjugacy will do.

*Project partially supported by national NSF of China (No 11171202).

There are recently a great deal of developments about S -conjugacy; we indicate briefly a few of them. Kudryavtseva investigated this conjugacy in regular epigroups with many elegant results [10], and is one of the main references for this paper. Lallement studied S -conjugacy for free semigroups [13]. Ganyushkin, Kormysheva and Mazorchuk described S -conjugacy in the symmetric inverse monoid R_n , showing that two elements are S -conjugate if and only if they have the same stable rank and the restrictions to their stable images have the same cycle type [7, 8]. Kudryavtseva and Mazorchuk then characterized S -conjugacy in Brauer-type semigroups and semigroups of square matrices [11, 12].

After the first wave by Clifford [4, 5], Lallement and Petrich [14], Munn [19, 20, 21], Ponizovskii [23], and Preston [24, 25, 26], there were many new developments in representation theory of finite semigroups. Rhodes and Zalcstein [32] obtained explicit constructions of the irreducible representations. Bidigare *et al* [1] and Brown [2, 3] found applications of semigroup representations to random walks, and established connections to Solomon's descent algebra. Putcha [28, 29] studied the representation theory of arbitrary finite monoids and determined all the irreducible characters of full transformation semigroups. Using Harish-Chandra's theory of cuspidal representations of finite groups of Lie type, Okninski and Putcha [22] showed that every complex representation of a finite monoid of Lie type is completely reducible.

Solomon [33] reformulated the Munn theory and determined the irreducible representations of R_n , with useful applications, by introducing central idempotents in the monoid algebra FR_n where F is a field of characteristic 0. Quite recently in [16, 17], this theory has been generalized to any Renner monoid. It turns out that the irreducible representations of the Renner monoid are completely determined by those of the parabolic subgroups of the Weyl group, and the number of inequivalent irreducible representations of the Renner monoid is the same as the number of S -conjugacy classes of the monoid.

Steinberg [35, 36] went even further, generalizing Solomon's approach to any finite inverse semigroup S . Using the Möbius function on S , Steinberg found the decomposition of the monoid algebra into a direct sum of matrix algebras over group rings and obtained the character formula for multiplicities. His formula is versatile in that he gave applications to decomposing tensor powers and exterior products of rook matrix representations in a more general setting.

The representation story is long, but one part is missing: whether there is a bijection between the irreducible representations of a finite inverse semigroup S over an algebraically closed field and S -conjugacy classes? The work of Kudryavtseva [10] on

S -conjugacy in regular epigroups and that of Steinberg [35, 36] on the representations of finite inverse semigroups lead the author to give an affirmative answer.

The organization of the paper is as follows. Section 2 provides necessary facts and background information on S -conjugacy in a semigroup. Section 3 is the main section and establishes a bijection between S -conjugacy classes of a finite inverse semigroup and the irreducible representations of S via the usual group conjugacy classes of all the maximal subgroups of S .

2 Preliminaries

Let S be a semigroup and $a \in S$. Denote by D_a and H_a the \mathcal{D} -class and \mathcal{H} -class at a in S , respectively (see standard textbooks [6] or [9] for Green relations). An element $a \in S$ is a group-bound element if there exists a positive integer k such that a^k lies in a subgroup of S . If every element of S is group-bound, we call S an epigroup, which is also named as a group-bound semigroup or strongly π -regular semigroup in the literature. Every finite semigroup is an epigroup; so is the full matrix monoid consisting of all square matrices over a field. Let $a \in S$ be group-bound such that H_{a^k} is a group whose identity element is denoted by e_a . It follows from Lemma 1 of [10] that the identity element e_a is well-defined. We call the element e_a the *idempotent induced from a* and the element ae_a the *invertible part of a* .

An element a of a semigroup S is called regular if there exists b in S such that $aba = a$. The semigroup S is called regular if all its elements are regular. The elements $a, b \in S$ are referred to as mutually inverse if $a = aba$ and $b = bab$. The following results taken from [10] are key in our discussion.

Theorem 2.1 *Let S be a semigroup and $a, b \in S$.*

- (a) *The invertible part ae_a of a is a group element and \mathcal{H} -related to e_a , and $ae_a = e_a a$.*
- (b) *If a and b are group elements and $a \sim b$, then $a\mathcal{D}b$ and $a \sim_p b$.*
- (c) *If a and b are group elements with $a\mathcal{H}b$ and $a \sim_p b$, then there exists $h \in H_{e_a}$ such that $a = h b h^{-1}$.*
- (d) *If S is regular and a is group-bound, then $a \sim ae_a$.*
- (e) *If S is a regular epigroup, then $a \sim b$ if and only if $ae_a \sim be_b$ if and only if there exists a pair of mutually inverse elements $u, v \in S$ such that $ae_a = u(be_b)v$ and $be_b = v(ae_a)u$.*

2.1 Inverse semigroups

A semigroup S is an inverse semigroup if each element $a \in S$ has a unique inverse $a^{-1} \in S$. It follows from the Vagner-Preston theorem that an inverse semigroup can be embedded faithfully into a symmetric inverse monoid I_X on a set X (see Theorem 5.1.7 of [9]), where I_X consists of all partial permutations of X . A partial permutation of X is a bijection $a : I \rightarrow J$ with $I, J \subseteq X$. The sets I and J are called the domain and range of a , respectively.

The binary product of two partial permutations in I_X is the usual composition of partial functions. The identity permutation, denoted by 1 , is the identity element of I_X , and the empty partial permutation, denoted by 0 , is the zero element. The unit group of I_X consists of full permutations of X , this group is isomorphic to the symmetric group on X .

If $X = \{1, 2, \dots, n\}$, we write R_n for I_X . Notice that R_n is called the rook monoid of size n in combinatorics. Indeed, R_n can be identified with the set of zero-one matrices which have at most one entry equal to 1 in each row and column. For instance with $\mathbf{n} = \{1, 2, 3, 4\}$, the following partial permutation

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & 1 & 2 & 3 \end{pmatrix}$$

corresponds to the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which has a one in position i, j if $a(j) = i$ and zero otherwise. The domain of this partial permutation is $\{2, 3, 4\}$ (column indices of 1's in the matrix) and range $\{1, 2, 3\}$ (row indices of the 1's).

We gather some basic properties and notation from Lawson [15] and Steinberg [36]. If $\sigma \in R_n$, denote by $\text{dom}(\sigma)$ the domain of $\sigma \in R_n$ and $\text{ran}(\sigma)$ the range. Then

$$\sigma^{-1}\sigma = 1_{\text{dom}(\sigma)}$$

$$\sigma\sigma^{-1} = 1_{\text{ran}(\sigma)},$$

where σ acts on the left of \mathbf{n} . Embedding an inverse semigroup S into R_n via rook matrices, we can regard $a^{-1}a$ the domain of a and aa^{-1} the range of a . This allows us to write

$$a^{-1}a = \text{dom}(a)$$

$$aa^{-1} = \text{ran}(a),$$

where we identify a partial identity on a subset with the subset itself. Furthermore, this gives us the freedom to think of a as a bijection from $\text{dom}(a)$ to $\text{ran}(a)$.

Denote by $E(S)$ the set of idempotents of S , which are crucial in determining the structure and representations of S . Define for every $e \in E(S)$

$$G(e) = \{a \in S \mid \text{dom}(a) = e = \text{ran}(a)\}.$$

Then $G(e)$ is a permutation group with the identity element e , and it is the same as the \mathcal{H} -class H_e , the maximal subgroup of S at e .

If $e, f \in S$ are \mathcal{D} -related, it follows from Proposition 2.3.5 of [9] that there exists an element $t \in S$ such that $\text{dom}(t) = f$ and $\text{ran}(t) = e$. Fix such t . Then the following mapping given by

$$\sigma_t : a \mapsto tat^{-1} \quad \text{for all } a \in G(f) \tag{1}$$

is a group isomorphism of $G(f)$ onto $G(e)$.

3 S -conjugacy and representations

From now on, we assume that S is a finite inverse semigroup, though some of the following arguments are valid for infinite inverse semigroups. Moreover, the S -conjugacy class of a in S will be denoted by $[a]$.

Let Λ be a set of idempotents of S such that the \mathcal{D} -classes $\{D_e \mid e \in \Lambda\}$ are a partition of S . Fix such Λ . Then

$$S = \bigsqcup_{e \in \Lambda} D_e.$$

Definition 3.1 *Let a be an element of S . If its invertible part ae_a lies in D_e for some $e \in \Lambda$, then e is called the subrank of a .*

Lemma 3.2 *Let $a \in S$.*

- (a) *All elements in $[a]$ have the same subrank.*
- (b) *If a has subrank $e \in \Lambda$ and e_a is the idempotent induced from a , then there exists $t \in S$ such that $t^{-1}t = e_a$ and $tt^{-1} = e$. Furthermore, $tat^{-1} \in [a]$.*

Proof. The proof of (a) is straightforward by Theorem 2.1 (b) and (e). As for (b), by Theorem 2.1 (a) we see that the invertible part ae_a of a is \mathcal{H} -related to the idempotent e_a induced from a , so $ae_a \in D_{e_a}$. But $ae_a \in D_e$ by the assumption. Thus $e_a \in D_e$. It follows from Proposition 2.3.5 of [9] that there is $t \in S$ such that $t^{-1}t = e_a$ and $tt^{-1} = e$, which

imply that $te_at^{-1} = e$ and $t^{-1}et = e_a$. Write $b = tat^{-1}$. Then the idempotent e_b induced from b is equal to e . In addition, $be_b = (tat^{-1})(te_at^{-1}) = ta_e at^{-1}$ and $ae_a = t^{-1}be_bt$. In view of Theorem 2.1 (e), we deduce that $tat^{-1} \in [a]$. \square

However, $t[a]t^{-1} \neq [a]$ in general; it can even happen that $t[a]t^{-1} \not\subseteq [a]$. For example, in the rook monoid R_3 , let $a = (32)[1]$. Then $[a] = \{(12)[3], (32)[1], (31)[2]\}$, where the notation $(ij)[k]$ means the partial permutation

$$\begin{pmatrix} i & j & k \\ j & i & - \end{pmatrix}$$

for $1 \leq i, j, k \leq 3$ and no two of them being the same. Fix $\Lambda = \{0, e_1, e_2, 1\}$, where $e_1 = \text{diag}\{1, 0, 0\}$ and $e_2 = \text{diag}\{1, 1, 0\}$. The subrank of $[a]$ is e_2 . But $e_a = a^2 = \text{diag}\{0, 1, 1\}$. Take

$$t = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in R_3$$

with $\text{dom}(t) = \{2, 3\}$ and $\text{ran}(t) = \{1, 2\}$. So $t = [321]$ and $t^{-1} = [123]$. A simple calculation yields that the element $t((31)[2])t^{-1} = 0 \notin [a]$.

Lemma 3.3 *Each S -conjugacy class $[a]$ in S meets a unique maximal subgroup $G(e)$ with $e \in \Lambda$. More specifically, e is the subrank of a .*

Proof. Let $e \in \Lambda$ be the subrank of a . The results of Theorem 2.1 (b) and (d) imply that $ae_a \in [a] \cap G(e_a)$. Since $e_a \in D_e$, there exists an element $t \in S$ such that $\text{dom}(t) = e_a$ and $\text{ran}(t) = e$. The conjugation by t is a group isomorphism of $G(e_a)$ onto $G(e)$. Observe that $tat^{-1} = ta_e at^{-1}$. Thanks to Lemma 3.2 (b) we obtain that $t(ae_a)t^{-1} \in [a] \cap G(e)$. The uniqueness of $G(e)$ follows from the fact that all elements of $[a]$ have the same subrank. \square

Proposition 3.4 *Let $a \in S$ have subrank $e \in \Lambda$. Then $[a] \cap G(e) = \overline{\sigma_t(ae_a)}$, the usual group conjugacy class of $\sigma_t(ae_a)$ in $G(e)$, where σ_t is defined as in (1).*

Proof. We first show that $[a] \cap G(e_a) = \overline{ae_a}$, the usual group conjugacy class of ae_a in $G(e_a)$. Clearly $\overline{ae_a} \subseteq [a] \cap G(e_a)$. We now prove the reverse inclusion. Let $x \in [a] \cap G(e_a)$. Note that x and ae_a are group elements in $G(e_a)$, and x is \mathcal{H} -related to ae_a . It follows from Theorem 2.1 (b) that $x \sim_p ae_a$. By Theorem 2.1 (c) there is an element $g \in G(e_a)$ for which $x = g(ae_a)g^{-1}$. Thus, $x \in \overline{ae_a}$.

Let $b = tat^{-1}$. A similar argument to that of Lemma 3.2 (b) shows that the invertible part of b is be , where e is the subrank of a . Then $[b] \cap G(e) = \overline{be}$, the usual group conjugacy class of be in $G(e)$. Thus $[a] \cap G(e) = \overline{be}$, since $[b] = [a]$. On the other hand, it is a simple matter that $t(\overline{ae_a})t^{-1} = \overline{be}$. Therefore, $[a] \cap G(e) = \overline{\sigma_t(\overline{ae_a})}$. \square

Theorem 3.5 *There is a bijection between the set of S -conjugacy classes of a finite inverse semigroup S and the set of group conjugate classes of the maximal subgroups $G(e)$ of S for $e \in \Lambda$.*

Proof. It follows from Proposition 3.4 that the conjugate classes of S that meet $G(e)$ are indexed by conjugate classes of $G(e)$ for $e \in \Lambda$. Thanks to Lemma 3.3 and $S = \bigsqcup_{e \in \Lambda} D_e$, the proof is complete. \square

Corollary 3.6 *The number of S -conjugacy classes of a finite inverse semigroup equals the sum of the numbers of group conjugate classes in $G(e)$ with $e \in \Lambda$.* \square

We now describe representations of a finite inverse semigroup S over a field F . Let n_e be the number of idempotents in D_e . It follows from Theorems 4.5 and 4.6 of [36] that

$$FS = \bigoplus_{e \in \Lambda} FD_e$$

and there exists an isomorphism ψ_e of FD_e onto $M_{n_e}(FG(e))$. Extend ψ_e to an algebra homomorphism of FS onto $M_{n_e}(FG(e))$ by $\psi_e(FD(f)) = 0$ for $f \in \Lambda \setminus \{e\}$. If $a \in FS$ then

$$\psi_e(a) = \sum_{i,j=1}^{n_e} \beta_{ij}(a) E_{ij}, \quad (2)$$

where $\beta_{ij}(a) \in FG(e)$ and E_{ij} are the matrix units for $1 \leq i, j \leq n_e$.

Applying ρ to the matrix entries of $\psi_e(a)$, we obtain that a representation ρ of $FG(e)$ gives rise to a representation ρ^* of FS by

$$\rho^*(a) = \sum_{i,j=1}^{n_e} \rho(\beta_{i,j}(a)) E_{i,j} \quad (3)$$

with $\deg \rho^* = n_e \deg \rho$. We identify a representation of S with its F -linear extension to a representation of the semigroup algebra FS and make a similar convention for representations of $G(e)$ and $FG(e)$.

Lemma 3.7 *Let F be a field, S a finite inverse semigroup, and $\widehat{FG}(e)$ a full set of inequivalent irreducible representations of $FG(e)$. If the characteristic of F is 0 or a prime number not dividing the order of any maximal subgroup of S , then $\{\rho^* \mid \rho \in \widehat{FG}(e), e \in \Lambda\}$ is a full set of inequivalent irreducible representations of FS .*

Proof. The result follows from Lemma 2.22 of [33] with a little modification. \square

We can now state our main result describing the relationship between S -conjugacy classes in S and representations of S over an algebraically closed field.

Theorem 3.8 *Let F be an algebraically closed field and S a finite inverse semigroup. If the characteristic of F is 0 or a prime number not dividing the order of any maximal subgroup of S , then the set of inequivalent irreducible representations of S over F is in bijection with the set of S -conjugacy classes in S .*

Proof. Lemma 3.7 shows that the irreducible representations of S are in bijection with the irreducible representations of maximal subgroups $G(e)$ where $e \in \Lambda$. On the other hand, from group representation theory, the irreducible representations of $G(e)$ are in bijection with the group conjugate classes of $G(e)$, since F is algebraically closed. The desired result follows from Theorem 3.5. \square

Corollary 3.9 *With the assumption and notation in Theorem 3.8, the number of inequivalent irreducible representations of a finite inverse semigroup S over F equals the number of S -conjugacy classes in S .*

Acknowledgments

The author would like to thank Dr. Reginald Koo for his useful suggestions.

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